

Effects of Topology in the Dirac Spectrum of Staggered Fermions *

F. Farchioni

Institut für Theoretische Physik,
Universität Bern, CH-3012 Bern, Switzerland

I. Hip and C. B. Lang

Institut für Theoretische Physik,
Universität Graz, A-8010 Graz, Austria

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Abstract

We compare the lower edge spectral fluctuations of the staggered lattice Dirac operator for the Schwinger model with the predictions of chiral Random Matrix Theory (chRMT). We verify their range of applicability, checking in particular the rôle of non-trivial topological sectors and the flavor symmetry of the staggered fermions for finite lattice spacing. Approaching the continuum limit we indeed find clear signals for topological modes in the eigenvalue spectrum. These findings indicate problems in the verification of the chRMT predictions.

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1 Introduction

Leutwyler and Smilga's sum rules [1] for the eigenvalues of the chiral Dirac operator for finite volume QCD suggest that the statistical fluctuations of the spectrum of this operator are universal in the infrared region; these should be described by a chiral Random Matrix Theory (chRMT) [2]. (For a recent review on Random Matrix Theory in general cf. [3].) Universality has been proved extensively in chRMT; for the case of QCD, see [4]. Explicit lattice calculations may provide an answer to the question, whether QCD lies really inside one of the universality classes of chRMT.

Chiral symmetry, which is the main ingredient here, may be effectively realized on the lattice by the so-called “overlap fermions” [5] and by the Fixed-Point (FP) fermion action [6], both satisfying the Ginsparg-Wilson [7] condition. Ginsparg-Wilson fermions – in spite of their theoretical appeal – present however non-trivial technical difficulties in a dynamical set-up (for first experiences in this context, see [8, 9]).

Kogut-Susskind “staggered” fermions, which are the object of this letter, realize chirality only partially, but are also much simpler, and still represent the typical framework for lattice QCD calculations when chirality is relevant. However, this is the main conclusion of this letter, comparison with chRMT is in this case not straightforward and subject to additional restrictions (besides the usual ones well-known in the literature) applying as long as the lattice cut-off is finite.

A class of problems comes from topology: when comparing lattice eigenvalue spectra with chRMT predictions, the validity of the index theorem [10] for the Dirac operator is assumed, while the staggered Dirac operator has no exact zero modes and a sound fermionic definition of the topological charge is not possible. Another class of problems is due to the flavor structure of staggered fermions, which is different at finite cut-off and in the continuum limit. Each copy of staggered fermions describes in the continuum $N_f = 2^{d/2}$ degenerate massless physical fermions; at finite cut-off the full chiral symmetry is broken into an abelian subgroup resembling the non-degenerate ($N_f = 1$) situation, with the relevant difference that the symmetry is here *anomaly-free*.

We study these issues in the simplified framework of the Schwinger model in the quenched and the dynamical situation. A particularity of two dimensions is that a continuous symmetry cannot be broken by the vacuum [11]. Since the chiral symmetry at finite cut-off is non-anomalous though abelian,

the usual fermion condensate vanishes even if staggered fermions act as just one flavor when comparing Dirac eigenvalue spectra with chRMT. With vanishing condensate, a different kind of universality should apply in the infrared part of the spectrum [12].

In a Monte Carlo simulation, all this applies of course only for the unquenched (dynamical) data. For the “quenched” model (corresponding to $N_f = 0$) the usual chRMT universality is expected to hold [8]. In this case, however, different complications come into play when the thermodynamic limit of the theory is taken [13]. Some results concerning the quenched data have been discussed in [14].

2 Predictions from Random Matrix Theory

The lower edge of the spectrum of the Dirac operator (i.e. its infra-red region) is expected to be universal at the scale of the level spacing $\Delta\lambda$ (microscopic scale) $\sim 1/V$. The ‘microscopic’ scaling variable is $z = V\Sigma\lambda$, where

$$\Sigma \equiv \pi \lim_{\lambda \rightarrow 0} \lim_{V \rightarrow \infty} \rho(\lambda) , \quad (1)$$

and $\rho(\lambda)$ denotes the associated spectral density per unit volume. When comparing the statistical properties of the Dirac operator with chRMT, Σ is an external scale parameter related to the dynamics of the physical system under study. The Banks-Casher formula [15] implies $\Sigma = -\langle \bar{\psi}\psi \rangle$.

Three classes of universality are predicted by chRMT [16], corresponding to (chiral) orthogonal, unitary and symplectic ensembles (chOE, chUE and chSE respectively). The universal properties considered in this letter are the microscopic spectral density

$$\rho_s(z) = \lim_{V \rightarrow \infty} \frac{1}{\Sigma} \rho\left(\frac{z}{V\Sigma}\right) \quad (2)$$

and the probability distribution of the smallest eigenvalue $P(\lambda_{\min})$. These distributions depend on the topological charge ν of the background gauge configuration and on the number of flavors N_f of quarks included in the dynamics. We also checked the level spacing statistics [14].

3 Statistical properties of the spectrum

Staggered fermions partially realize chiral symmetry on the lattice. One species of staggered fermion describes $N_f = 2^{d/2}$ Dirac fermions in the continuum limit. For finite lattice spacing the chiral symmetry $U_V(2^{d/2}) \times U_A(2^{d/2})$ is however broken down to a $U_V(1) \times U_A(1)$ *non-anomalous* sub-symmetry, which accounts for the reflection-invariance for the purely imaginary spectrum.

In chRMT, the number of dynamical fermions N_f and the topological charge ν of the background configuration enter as external parameters. Our (compact) gauge configurations were sampled in the quenched set-up, i.e. according to the pure gauge measure (given by the standard plaquette action). This corresponds in chRMT to the case $N_f = 0$. Assuming no extra symmetry for the Dirac operator (besides the one mentioned above) in our two-dimensional set-up, the statistical fluctuations of the spectrum in the infrared region are expected [16] to be described by chUE; this was confirmed by previous results [8] with the Fixed Point and Neuberger's overlap action, where however, deviations (i.e. a distribution resembling chSE) were observed whenever the physical volume was too small.

In order to explore the dynamics of fermions, we then re-weighted the quenched configurations with the fermion determinant. For $d = 4$ this procedure usually introduces large statistical fluctuations; in our two-dimensional model these fluctuations did not appear to be too harmful [8]. Since the chiral symmetry of lattice staggered fermions is abelian, one expects [17] that the dynamical situation is described by a chRMT with $N_f = 1$. The two-dimensional situation is however peculiar due to the Mermin-Wagner-Coleman theorem stating that a continuous symmetry cannot be broken by the vacuum; only an anomaly can do that. Since the operator $\bar{\psi}(x)\psi(x)$ is not invariant under the residual (non-anomalous) abelian chiral symmetry of staggered fermions, its expectation value vanishes [18]. (One can argue [19] that the simplest observable signaling the breaking of the *anomalous* axial symmetry is in this context $\langle \bar{u}u\bar{d}d \rangle$.) With vanishing fermion condensate, the universality class of chUE does not apply anymore, and a new kind of universality holds [12].

The validity of the $N_f = 1$ chRMT predictions is restricted. In the continuum limit the irrelevant terms of the action breaking the full symmetry vanish and a transition to $N_f = 2$ is due to occur with doubling of the spectrum. In this limit, two degenerate fermions described by two-component spinors are

the effective degrees of freedom, the two-flavor (massless) Schwinger model being the underlying continuum theory [18].

RMT predictions depend also on the topological charge of the background gauge configuration. In the continuum the topological charge is related to the index of the Dirac operator [10] (the number of zero modes counted according to their chirality). On the lattice, a consistent definition of the topological charge in terms of the index of the Dirac operator is possible only for Ginsparg-Wilson (i.e. Overlap and FP) fermions [20]. For staggered fermions, away from the continuum limit, the Dirac operator has no exact zero modes, and the naive expectation is that all configurations show spectral distributions described by chRMT predictions for the trivial sector [17, 21].

However, for lattices that are fine enough one should attain the continuum situation. We will explore this hypothetical scenario and check possible systematic effects when comparing the spectrum of the Dirac operator with chRMT predictions. Lacking (in this case) a consistent fermionic definition of the topological charge ν , we rely on the geometric definition of the gauge configurations. In the analysis we distinguish between the sectors with trivial ($\nu = 0$) and non-trivial topological charge.

3.1 The simulation

We considered statistically independent gauge configurations for lattices of size 16^2 ($\beta = 2, 4, 6$; 10000 configurations); 24^2 ($\beta = 2, 4, 6, 8$; 5000 configurations) and 32^2 ($\beta = 2, 4, 6$; 5000 configurations). Even-odd preconditioning and standard LAPACK routines were used to diagonalize the staggered Dirac operator.

The trivial sector. Fig. 1 shows the distribution of the smallest eigenvalue $P(\lambda_{\min})$ for some lattice volumes. The histograms are obtained by choosing configurations with geometrical topological charge $\nu = 0$; this reduces the statistics to 30% – 10% of the original one depending on L and β .

We observe consistency with chUE except for $L = 16$, $\beta \geq 6$, where the physical volume is likely to be too small for chRMT to apply. For the fixed point and Neuberger’s action an apparent transition to chSE, not observed here, took place for small lattices and/or high values of β [8].

The fit of the chUE distribution provides an estimate for Σ which is given in Table 1 and in Fig. 5, left (we are using $ea = 1/\sqrt{\beta}$ as abscissa, in the

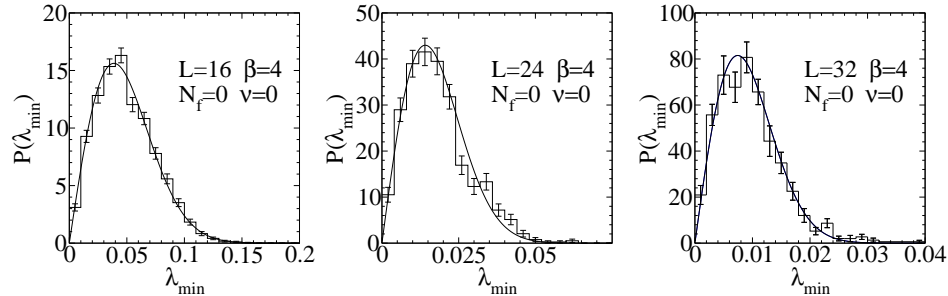


Figure 1: Distribution of the smallest eigenvalue for different lattice sizes and $\beta = 4$ (quenched case); the continuous curve represents the theoretical prediction of chUE.

ordinate we report $\Sigma\sqrt{\beta} = -\langle\bar{\psi}\psi\rangle_{\text{cont}}/e$; the equations hold in the weak coupling limit of the Schwinger model with one flavor).

Following RMT the spectrum is to be separated in a fluctuation part (conjectured to follow chRMT predictions) and a smooth background. The information on the chiral condensate is contained in both. Removing the smooth background is done by “unfolding”: the eigenvalues are mapped to a variable z , such that the *average* level spacing is constant. In Fig. 2 we plot the microscopic spectral density $\rho_s(z)$ for the unfolded eigenvalues for two situations. In the unfolding process the average value of the spectral density near $\lambda = 0$ provides a direct estimate for Σ (see (1)). Neglecting finite-size effects, this value should agree with the number obtained from the smallest eigenvalue distribution in the scaling region of chRMT. We find discrepancies between the two determinations for $\beta \geq 4$, the “background” value of Σ (i.e. $\pi\rho(0)$) coming out larger; as a consequence of this discrepancy, the spectral density does not follow the predicted shape anymore (Fig. 2, right). We interpret this as due to the transition to the full flavor-symmetric situation of the continuum limit, which in the quenched case implies trivial doubling of the spectrum.

It has been suggested [13] that the spectral density of the quenched Schwinger model should diverge exponentially with the volume for small eigenvalues; our data do not allow us to draw any firm statement concerning this. We only observe that for $\beta=2$ and 4, $L=24$ and 32 give consistent values of Σ , while it is no more the case for the larger value of $\beta=6$, this indicating a slowing down of the scaling behavior in the weak coupling region.

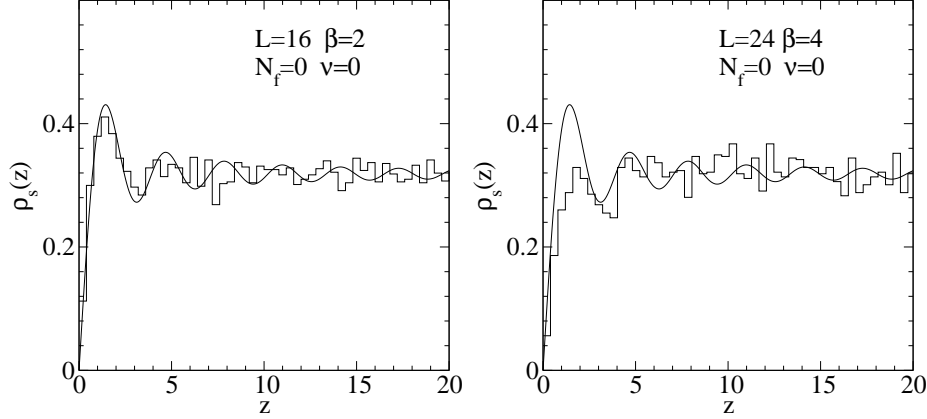


Figure 2: Microscopic spectral density $\rho_s(z)$ for different lattices and values of β (quenched case); the continuous curves represent the theoretical prediction of chUE.

Topological sector. Fig. 3 shows the distributions of the first and second eigenvalue for configurations with $|\nu| = 1$. Ideally, the first eigenvalue ought to be a zero mode, while the second eigenvalue should follow the predictions of chUE for the smallest eigenvalue for $|\nu| = 1$; these may be obtained just using the values of Σ found in the trivial sector in the formulas for the smallest eigenvalue distribution with $|\nu| = 1$ (the continuous line in the Fig. 3).

What we see instead is that for small β the first eigenvalue follows the chUE statistics of the smallest eigenvalue in the trivial sector (dashed curve in the figure). Such a behavior was also observed for $SU(3)$, $d=4$ [21]. Increasing β , the first eigenvalue peaks closer and closer to zero and the *second smallest* eigenvalue approaches the predictions of chRMT for the *smallest* eigenvalue for $|\nu| = 1$ ($\beta \geq 4$). We conclude that only towards the continuum limit the zero modes correlate clearly with the topological sector of the gauge configuration.

In order to check the systematics of the possible misidentification, we repeated the fits with chRMT shapes *without* selecting the trivial topological sector: even for the smallest value of β at our disposal, $\beta = 2$, we always obtain considerably larger values of $\chi^2/d.o.f.$ (data with asterisk in Table 1) and the fitting values of Σ are inconsistent with the determination in the trivial sector. We conclude, that even for moderate values of β , the identification of the topological sector is an essential ingredient for the correct

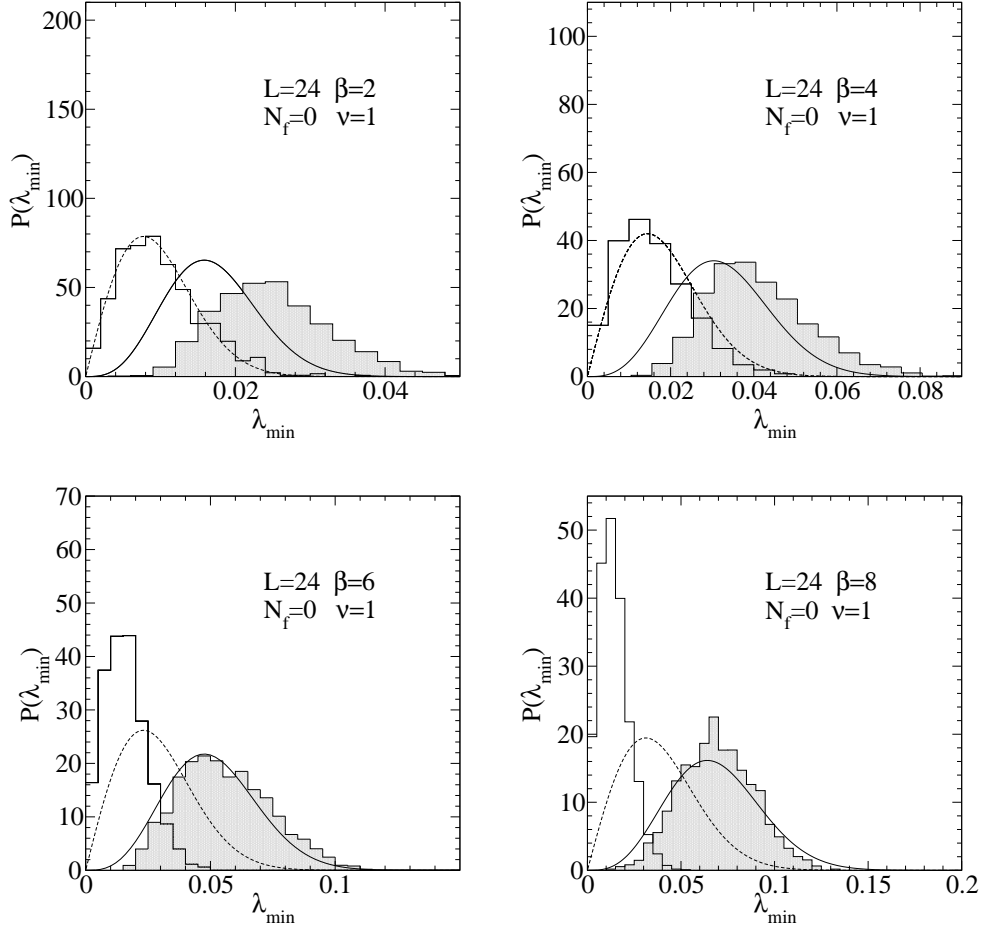


Figure 3: Distribution of the smallest and second (shaded histogram) eigenvalue for $|\nu| = 1$. The continuous (dashed) curves represent the theoretical prediction of chUE for the smallest eigenvalue for $|\nu| = 1$ ($\nu = 0$) based on the Σ -value from the fit to the $\nu = 0$ data discussed earlier.

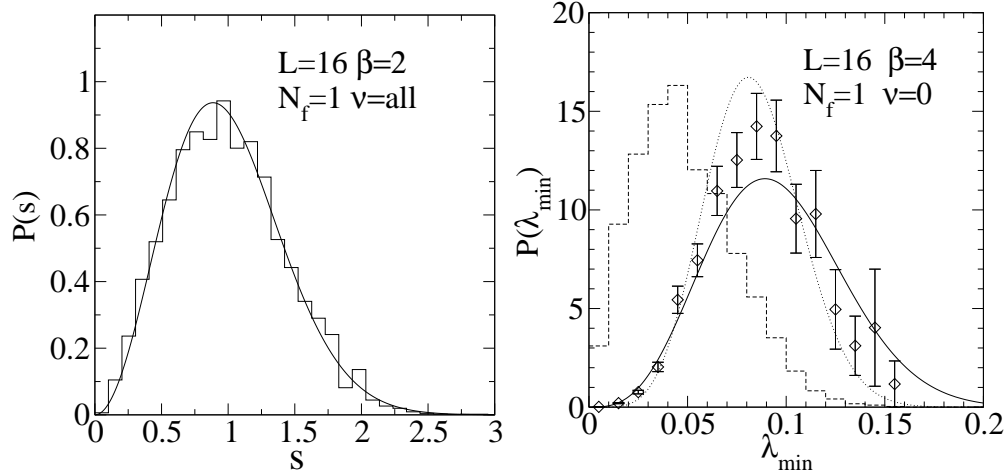


Figure 4: Results for eigenvalues after re-weighting with the fermion determinant, corresponding to dynamical fermions. Left: Level spacing distribution (unfolded variable), compared with the theoretical expectation for chUE[3]. Right: Distribution of the smallest eigenvalue (diamonds with error bars). The full and dotted curves are the best fits from chUE with $N_f = 1$ and $N_f = 2$ respectively, resulting in Σ values as given in Table 1. For comparison we report also the histogram of quenched data (dashed line).

comparison with chRMT.

Re-weighting. Of high interest in chRMT is the dynamics of fermions. In order to account for dynamical fermions, we re-weight the quenched gauge configurations with the fermion determinant; in the case of the statistics $P(\lambda_{\min})$, for example, this amounts to the following redefinition:

$$P(\lambda_{\min}) = \frac{\sum_C \delta(\lambda_{\min} - \lambda_{\min}(C)) \det \mathcal{D}(C)}{\sum_C \det \mathcal{D}(C)} , \quad (3)$$

where $\lambda_{\min}(C)$ denotes the smallest eigenvalue for the configuration C . The fluctuation of the determinant introduces additional fluctuations in the studied quantity (in our case the error typically doubles), which can be traced back to the effective loss of statistics of the gauge sample. Loss of statistics is also caused by the depletion of the trivial sector (the one under study) on large lattices.

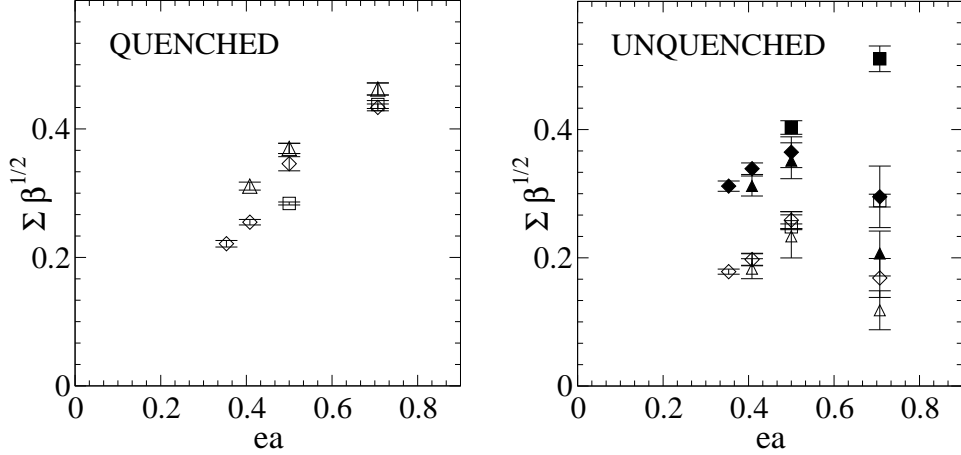


Figure 5: The condensate Σ as obtained from the fits on the smallest eigenvalue distribution using chRMT predictions, for the quenched (left) and re-weighted (right) gauge configurations: $L = 16$ (boxes), $L=24$ (diamonds), $L=32$ (triangles); in the dynamical case we plot the fit values obtained from comparison with chUE for $N_f = 1$ (open symbols) and $N_f = 2$ (full symbols).

As already discussed in [14], in the quenched situation we find perfect chUE behavior for the level spacing distribution for all parameter values studied. This observation persists also for the re-weighted, unquenched data (e.g. Fig. 4 left). For the level spacing we always find chUE distribution. Such a behavior in a situation of vanishing fermion condensate was found already in the Coulomb phase of compact four dimensional QED [22].

In Fig. 4 (right) we report the distribution of the smallest eigenvalue for $L = 16$, $\beta = 4$ and find it clearly different from the quenched distribution. As expected from the discussion of Sec. 3 neither the $N_f = 1$ chUE shape nor the $N_f = 2$ succeed in fitting the distribution. The two different fits result in different values of Σ reported in Table 1 and Fig. 5, right. In both cases the values of $\chi^2/d.o.f.$ of the fit are rather large, confirming that chUE does not give the right prediction. The fitted values of Σ are well above zero. As discussed, strictly speaking Σ ought to be zero, a different chRMT and universality class applying. In the (true) one flavor situation the continuum condensate is $-\langle\bar{\psi}\psi\rangle_{\text{cont}}/e \approx 0.15989$; this situation is described by one copy of Overlap or FP fermions [8].

	L	β	quen. ($N_f = 0$)	rew. ($N_f = 1$)	rew. ($N_f = 2$)
	16	2	0.3282(31) [0.81]	0.2058(72) [0.56]	0.363(14) [1.05]
*)	16	2	0.3126(34) [3.26]		
	24	2	0.3086(38) [0.65]	0.120(22) [0.26]	0.210(34) [0.28]
*)	24	2	0.3264(36) [2.44]		
	32	2	0.3289(67) [0.72]	0.084(21) [1.41]	0.149(26) [1.40]
*)	32	2	0.3319(48) [3.84]		
	16	4	0.1420(12) [0.83]	0.1256(22) [1.97]	0.2026(54) [5.25]
	24	4	0.1741(37) [1.42]	0.1302(67) [0.79]	0.184(12) [1.99]
	32	4	0.1860(40) [1.21]	0.118(17) [0.43]	0.178(14) [0.20]
	24	6	0.1047(18) [1.25]	0.0811(38) [4.83]	0.1391(38) [2.58]
	32	6	0.1271(25) [1.37]	0.0749(64) [3.93]	0.1273(64) [2.74]
	24	8	0.0787(18) [3.17]	0.0635(22) [3.85]	0.1110(27) [2.89]

Table 1: Values of the fermion condensate obtained from comparison of lattice data with chRMT (chUE). In the square brackets we indicate $\chi^2/d.o.f.$. Data with asterisk are obtained without selecting a topological sector.

4 Discussion

We conclude that for a careful RMT analysis of the eigenvalue spectrum of the staggered Dirac operator the identification of the topological sector is advisable. Towards the continuum limit the topologically non-trivial sectors exhibit an extra peak at small eigenvalues, which we attribute to the “want-to-be” zero modes. For a recent discussion on the relevance of quasi-zero modes and their chirality in the context of staggered fermions cf. [23].

We identified problems due to the general ambiguity of staggered fermions between the choice $N_f = 1$ applying for finite lattice spacing and $N_f = 2^{d/2}$ applying in the continuum limit. For example, the quenched microscopic spectral density clearly deviates from chUE for $\beta \geq 4$, because the “background” estimate of Σ , $\pi\rho(0)$, does not match with the “microscopic” one as can be obtained e.g. from the smallest eigenvalue distribution.

Additional unexpected features come into play because of our two-dimensional context: the Mermin-Wagner-Coleman theorem implies a vanishing fermion condensate with the result that chUE fails in the case of dynamical fermions. This is indicated by our data, even if the statistics is not good enough to draw firm conclusions. This scenario is quite different from that found in [8] where the Schwinger model with Ginsparg-Wilson fermions was

analyzed; there, genuine single-flavored fermions were described.

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